The Skolem-Bang Theorems in Ordered Fields with an IP

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Abstract

This paper is concerned with the extent to which the Skolem-Bang theorems in Diophantine approximations generalise from the standard setting of $\langle \mathbb{R}, \mathbb{Z} \rangle$ to structures of the form $\langle F, I \rangle$, where F is an ordered field and I is an integer part of F. We show that some of these theorems are hold unconditionally in general case (ordered fields with an integer part). The remainder results are based on Dirichlet's and Kronecker's theorems. Finally we extend Dirichlet's theorem to ordered fields with IE_1 integer part.

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1 Introduction

Let $\alpha \geqslant 1$ be a real number. The notion of $\mathbb{N}_\alpha$ was introduced by Skolem and Bang as the sequence $\{\lfloor n\alpha \rfloor |\ n \in \mathbb{N}\ \}$ of positive integers, where $\lfloor x \rfloor$ is the integer part of x. The following facts are studied in Skolem-Bang Theorems [11, 2]:

1. $\mathbb{N}_\alpha \cap \mathbb{N}_\beta = \{0\}$; **2**. $\mathbb{N}_\alpha \cup \mathbb{N}_\beta = \mathbb{N}$; **3**. $\mathbb{N}_\alpha \subseteq \mathbb{N}_\beta$.

These theorems are also reported in [9]:

Fact A. Let α and β be positive real numbers. Then $\mathbb{N}_\alpha \cup \mathbb{N}_\beta = \{0\}$ and $\mathbb{N}_\alpha \cap \mathbb{N}_\beta = \{0\}$ if and only if α, β are irrational numbers and $\alpha^{-1} + \beta^{-1} = 1$.

Fact B. Let α and β be positive real numbers. If $1, \alpha^{-1}, \beta^{-1}$ are linearly independent over the field of rational numbers, then $\mathbb{N}_\alpha$ and $\mathbb{N}_\beta$ have infinitely many common elements.

Fact C. Let α and β be positive real numbers such that $a\alpha^{-1} + b\beta^{-1} = c$ for some integers a, b, c, with $ab < 0$ and $c \neq 0$. Then $\mathbb{N}_\alpha$ and $\mathbb{N}_\beta$ have infinitely many common elements.

Fact D. Let α and β be positive real numbers such that $a\alpha^{-1} + b\beta^{-1} = c$ for some positive integers a, b, c, with $(a, b, c) = 1$ and $c > 1$. Then $\mathbb{N}_\alpha$ and $\mathbb{N}_\beta$ have infinitely many common elements.

Fact E. Let α and β be positive real numbers. The sets $\mathbb{N}_\alpha$ and $\mathbb{N}_\beta$ are disjoint if and only if α and β are irrational numbers and there exist positive integers a and b such that $a\alpha^{-1}+b\beta^{-1} = 1$.

Further more if $\mathbb{N}_\alpha$ and $\mathbb{N}_\beta$ have one common element, they have infinitely many ones.

Fact F. Let α and β be positive irrational numbers. Then $\mathbb{N}_\alpha \supseteq \mathbb{N}_\beta$ if and only if there exist positive integers a and b such that $a(1 - \alpha^{-1}) + b\beta^{-1} = 1$.

The rational version of Fact F is the following:

Fact F'. Let σ and ρ be positive rational numbers. Then $\mathbb{N}_\sigma \supseteq \mathbb{N}_\rho$ if and only if there exist positive integers a and b such that $a(1 - \sigma^{-1}) + b\rho^{-1} = 1$.

There are also some other Skolem-Bang results which are either trivial or obtained from the above ones. All these results are based on two important theorems in the theory of Diophantine Approximations: *Dirichlet's Theorem* and *Kronecker's Theorem*.

Dirichlet's Theorem. Let θ be a positive irrational number. There are infinitely many rational numbers $\frac{a}{b}$, where a and b are positive integers, such that

$$|\theta - \frac{a}{b}| < \frac{1}{b^2} \ .$$

An immediate conclusion of Dirichlet's Theorem is that the set $\{n\theta - \lfloor n\theta \rfloor |\ n \in \mathbb{N}\}$ is a dense subset of $[0, 1)$. A more interesting corollary is

Separability property. Let $\alpha, \beta > 1$ be real numbers. Then $\alpha \neq \beta$ if and only if $\mathbb{N}_\alpha \neq \mathbb{N}_\beta$.

Kronecker's Theorem . Let α and β be positive irrationals such that $1, \alpha, \beta$ are linearly independent over the field of rational numbers, then the points whose coordinates are the fractional parts of multiples of α and β, i.e. $(n\alpha - \lfloor n\alpha \rfloor, n\beta - \lfloor n\beta \rfloor)$, $n = 1, 2, 3, \ldots$, are dense in the *unite square.*

Note that Fact B and Kronecker's Theorem are equivalent. Dirichlet's Theorem and Kronecker's Theorem are based on Pigeon Hole Principle (**PHP**) and Box Principle (the two dimensional version of Pigeon Hole Principle). However, in non-Archimedean cases, PHP and Box Principle do not hold. Extending the notion of the *separability property* to non-Archimedean structures $\langle F, I\rangle$ is a useful tool to generalize Skolem-Bang Theorems to these structures. If F is an ordered field and I an integer part for F, we call $\langle F, I\rangle$ *separable* if it satisfies the separability property. Mojtaba Moniri has conjectured that "*any arbitrary structure* $\langle F, I\rangle$ *is separable*". In Section 2, we prove some weak versions of the separability property, i.e. we prove it for the cases that:

1. α and β are irrationals;

2. α and β are rationals;

3. $\alpha, \beta \geqslant 2$;

4. ρ is a rational number and α is an irrational number such that $1 < \rho < \alpha < 2$.

In Section 3, we will show that any $\langle F, I\rangle$ satisfying Dirichlet's Theorem is separable and Fact A hold in separable $\langle F, I\rangle$. Also if $\langle F, I\rangle$ satisfies Dirichlet's Theorem and I is a Bézout domain then Facts C, D and one direction of Facts E, F hold. We also show that Fact F' holds for a structure $\langle F, I\rangle$ in which I is a Bézout domain.

The main tool of Section 4 is *Farey series* which is studied in Hardy and Wright's excellent book [6]. Using *weak* versions of **PHP**, we can prove some special forms of Dirichlet's Theorem in *weak* fragments of Arithmetic. In [5, Theorem 3.1], P. D'Aquino proved a weak version of Dirichlet's Theorem in $I\Delta_0 + \Omega_1$, where Ω_1 is

$$\forall x \exists y (x^{log(x)} = y)$$

and by $log(x)$ we mean the integer part of $\log_2(x)$, (for more details, see subsection 4.1). Using Farey series, we prove Dirichlet's Theorem in $\langle F, I\rangle$ in which I is an IE_1-model. Since Wilmers proved that $IE_1 \vDash B\acute{e}z$, Facts C, D and one direction of Facts E, F mentioned in Section 3 hold in IE_1, [13]. In [10], B. Segre provided an asymmetric Diophantine approximations theorem for irrational numbers. We prove this theorem by a similar method based on Farey series which represented in proof theorem 1.7 in [9]. This theorem has interesting corollaries such as Hurwitz

asymmetric Theorem which will be denoted in Section 4. Their proofs are similar to the real case.

It is not known whether Kronecker's Theorem holds for IE_1-models. If so, all the Skolem-Bang theorems hold in any $\langle F, I\rangle$ in which I is an IE_1-model. In fact the remainder direction of Facts E, F are based on Kronecker's theorem.

1.1 The Preliminaries

Let $L = \{+, -, \cdot, 0, 1, \leqslant\}$ be the language of ordered rings. We deal with the following sets of axioms in L:

DOR: discretely ordered rings i.e., axioms for ordered rings together with $\forall x \neg(0 < x < 1)$.

ZR: discretely ordered $\mathbb{Z}$-rings i.e., **DOR** together with the condition that for every $n \in \mathbb{N}^{\geqslant 2}$, we have $(\forall x)(\exists q, r)(x = nq + r \bigwedge 0 \leqslant r < n)$.

EDR: Euclidean division rings i.e., **DOR** extended with the scheme of axioms that for every $n \in I^{>0}$, $(\forall x)(\exists q, r)(x = nq + r \bigwedge 0 \leqslant r < n)$.

IOP: Open induction i.e., **DOR** plus the following scheme for open L-formulas $\varphi(x, y)$

$$\forall \vec{x}, (\varphi(\vec{x}, 0) \bigwedge \forall y \geqslant 0, (\varphi(\vec{x}, y) \rightarrow \varphi(\vec{x}, y+1)) \rightarrow \forall y \geqslant 0, \varphi(\vec{x}, y)).$$

We define the formula class $E_n, U_n, \forall_n, \exists_n$ in the usual way:

$$E_0 = U_0 = \{\phi(\bar{x}) : \phi \textit{ is open}\}$$

$$\exists_{n+1} = \{\exists \bar{y}\phi(\bar{x}, \bar{y}) : \phi \in \forall_n\},\ \forall_{n+1} = \{\forall \bar{y}\phi(\bar{x}, \bar{y}) : \phi \in \exists_n\}$$

$$E_{n+1} = \{\exists \bar{y} \leqslant t(\bar{x})\phi(\bar{x}, \bar{y}) : \phi \in U_n,\ t \textit{ a term in } L\}$$

$$U_{n+1} = \{\forall \bar{y} \leqslant t(\bar{x})\phi(\bar{x}, \bar{y}) : \phi \in E_n,\ t \textit{ a term in } L\}$$

$$\Delta_0 = \bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} U_n$$

$\mathbf{IE_1}$: Bounded existential induction i.e., **DOR** plus the induction schema for all E_1-formulas φ:

$$\forall \vec{x}, (\varphi(\vec{x}, 0) \bigwedge \forall y \geqslant 0, (\varphi(\vec{x}, y) \rightarrow \varphi(\vec{x}, y+1)) \rightarrow \forall y \geqslant 0, \varphi(\vec{x}, y))).$$

We can define $IE_n, IU_n, I\Delta_0$ similarly.

We say that a subring I of an ordered field F is an integer part (**IP**) of F if $I \vDash$ **DOR** and for every $x \in F$, there is $a \in I$ such that $a \leqslant x < a+1$. We call this unique element a the integer part of x and write $a = \lfloor x \rfloor_I$. Every real closed field has an **IP**, [8]. On the other hand, there exist ordered fields without any **IP**, (see [4, 7]). One can see that every **IP** is an **EDR** and every **EDR** is an **IP** for its fraction field.

We use $\langle F, I\rangle$, for an ordered field F equipped with an **IP** I. We set $Q = Frac(I)$, the fraction field of I. We say I is **Bézout** if for each $m, n \in I^{\neq 0}$, there exist $r, s \in I$ such that $rm + sn \geqslant 1$ and $rm + sn | m, n$. Thus $rm + sn$ is greatest common divisor of m, n.

2 The Skolem-Bang Integer Part Theorems

Skolem and Bang theorems (see [11, 2] and [9]), for the standard case, is based on very special properties of $\mathbb{R}$ and $\mathbb{Z}$, such as **PHP**. In this section, we deal with these theorems in our an arbitrary $\langle F, I\rangle$.

Fix $\langle F, I\rangle$ and let $m, k \in I^{\geqslant 0}$, with $0 \leqslant k < m$ and $\alpha \in F^{>0}$. We define an *arithmetical progression* by parameters m and k as follows, $mI^{\geqslant 0} + k = \{mt + k \mid t \in I^{\geqslant 0}\}$. As the classic case, let $N_\alpha = \{\lfloor n\alpha \rfloor_I \mid n \in I^{\geqslant 0}\}$ and $\alpha I^{\geqslant 0} = \{n\alpha \mid n \in I^{\geqslant 0}\}$. It is easy to verify N_α when $0 < \alpha < 1$.

Lemma 2.1 *Let $\alpha \in F$. Then $0 < \alpha \leqslant 1$ if and only if $N_\alpha = I^{\geqslant 0}$.*

Proof. (**Only if.**) Suppose $0 < \alpha < 1$ and pick an arbitrary $n \in I^{>0}$. Let $k = \lfloor \frac{n}{\alpha} \rfloor$. We have $k \leqslant \frac{n}{\alpha} < k+1$ and so $n < (k+1)\alpha \leqslant n + \alpha$ which is less than $n+1$. Therefore $\lfloor (k+1)\alpha \rfloor = n \in I^{>0}$.

(**If.**) Suppose $\alpha > 1$ and let $k = \lfloor \frac{1}{\alpha - 1} \rfloor$. Then $k \leqslant \frac{1}{\alpha-1} < k+1$. So

$$k < k\alpha \leqslant k+1 < k+2 < (k+1)\alpha.$$

We now distinguish two cases. If $k\alpha < k+1$, then $\lfloor k\alpha \rfloor = k$ and $\lfloor (k+1)\alpha \rfloor \geqslant k+2$. Therefore $k+1 \notin I_\alpha^{\geqslant 0}$.

If $k\alpha = k+1$, then $\alpha = \frac{k+1}{k}$. So, there is no $n \in I^{\geqslant 0}$ such that $\lfloor n\alpha \rfloor = k$. The reason is that $(k-1)\alpha = k+1-\alpha < k < k+1 = k\alpha$. ■

In the following theorem we show that when $\alpha > 1$ is a positive rational number, N_α is a union of some arithmetical progressions. Moreover, if $\alpha, \beta > 1$ are two distinct rational numbers, then $N_\alpha \cap N_\beta$ and $I^{>0} \setminus (N_\alpha \cup N_\beta)$ are cofinal subsets of $I^{>0}$.

Theorem 2.2 *We have*

1. *If $\alpha = \frac{p}{q} > 1$, $p, q \in I^{\geqslant 0}$ and $q \neq 0$, then $N_\alpha = \bigcup_{0 \leqslant r < q}(pI^{\geqslant 0} + \lfloor \frac{pr}{q} \rfloor)$ and*

$$(pI^{\geqslant 0} + (p-1)) \subseteq I^{\geqslant 0} \setminus N_\alpha.$$

2. *If $\alpha, \beta > 1$ are rationals, then $N_\alpha \cap N_\beta$ and $I^{\geqslant 0} \setminus (N_\alpha \cup N_\beta)$ are cofinal subsets of I.*

3. *If $\alpha_i > 1$, $i = 1, \cdots, n$ are rationals, then $\bigcap_{i=1}^{n} N_{\alpha_i}$ and $I^{\geqslant 0} \setminus \bigcup_{i=1}^{n} N_{\alpha_i}$ are cofinal in I.*

Proof. 1) Since I is an Euclidean division ring (**EDR**), for each $n \in I^{\geqslant 0}$ there exist $r, k \in I^{\geqslant 0}$ such that $0 \leqslant r < q$ and $n = kq + r$. Therefore $n\frac{p}{q} = (kq+r)\frac{p}{q} = kp + \frac{pr}{q}$. Then $\frac{pr}{q} \leqslant \frac{p(q-1)}{q} < p-1$. So, $N_\alpha = N_{\frac{p}{q}} = \bigcup^{\circ}_{0 \leqslant r < q}(pI^{\geqslant 0} + \lfloor \frac{pr}{q} \rfloor)$. If $r, s \in I^{\geqslant 0}$, where $0 \leqslant r < s < q$, then we have $\frac{ps}{q} - \frac{pr}{q} = \frac{p(s-r)}{q} \geqslant \frac{p}{q} > 1$. Therefore $\lfloor \frac{pr}{q} \rfloor \not\equiv \lfloor \frac{ps}{q} \rfloor (mod\ p)$, since $0 \leqslant \frac{pr}{q} < \frac{ps}{q} < p$. So, these arithmetical progressions are disjoint. The other arithmetical progressions modulo p appear in $I^{\geqslant 0} \setminus N_\alpha$ as $pI^{\geqslant 0} + (p-1)$.

2) Let $\alpha = \frac{p_1}{q_1}, \beta = \frac{p_2}{q_2}$, with $p_i, q_i \in I^{\geqslant 0}, i = 1, 2$. Then $p_1 p_2 I^{\geqslant 0} \subseteq N_\alpha \cap N_\beta$ and

$$(p_1 p_2 I^{\geqslant 0} + (p_1 p_2 - 1)) \cap (N_\alpha \cup N_\beta) = \emptyset.$$

3) Let $\alpha_i = \frac{p_i}{q_i}$, with $p_i, q_i \in I^{\geqslant 0}, i = 1, 2, ..., n$. Then we have $(\pi_{i=1}^{n} p_i) I^{\geqslant 0} \subseteq \bigcap_{i=1}^{n} N_{\alpha_i}$ and $[((\pi_{i=1}^{n} p_i) I^{\geqslant 0} + ((\pi_{i=1}^{n} p_i) - 1))] \cap (\bigcup_{i=1}^{n} N_{\alpha_i}) = \emptyset$. This completes the proof. ■

We have different situations for N_α with respect to rational and irrational elements when $\alpha > 1$. First we prove a basic property when $\alpha, \beta \geqslant 2$.

Theorem 2.3 *Let $\alpha, \beta \in F$ with $\alpha > \beta \geqslant 2$. Then $N_\beta \setminus N_\alpha \neq \emptyset$.*

Proof. Let $m = \lfloor \frac{1}{\alpha - \beta} \rfloor$. If $m = 0$, then $\alpha - \beta > 1$ and so $\lfloor \beta \rfloor \in N_\beta \setminus N_\alpha$. Otherwise, $m > 0$. We claim that $\lfloor (m+1)\beta \rfloor \in N_\beta \setminus N_\alpha$. First note that $m \leqslant \frac{1}{\alpha - \beta} < m + 1$ and therefore

$$m\beta < m\alpha \leqslant m\beta + 1 < m\beta + \beta = (m+1)\beta < \lfloor (m+1)\beta \rfloor + 1 < (m+1)\alpha.$$

We distinguish two cases.

Case (1). $\lfloor m\beta \rfloor = \lfloor m\alpha \rfloor$. In this case we have

$$\lfloor m\alpha \rfloor = \lfloor m\beta \rfloor < \lfloor (m+1)\beta \rfloor < \lfloor (m+1)\beta \rfloor + 1 \leqslant \lfloor (m+1)\alpha \rfloor.$$

This proves the claim.

Case(2). $\lfloor m\beta \rfloor \neq \lfloor m\alpha \rfloor$. In this case we have $\lfloor m\alpha \rfloor = \lfloor m\beta \rfloor + 1$. Therefore

$$\lfloor m\alpha \rfloor = \lfloor m\beta \rfloor + 1 < (m\beta + 1) + 1 \leqslant (m+1)\beta < \lfloor (m+1)\beta \rfloor + 1 \leqslant (m+1)\beta + 1 < (m+1)\alpha.$$

Hence $\lfloor (m+1)\beta \rfloor < \lfloor (m+1)\beta \rfloor + 1 \leqslant \lfloor (m+1)\alpha \rfloor$.

We show that $\lfloor m\alpha \rfloor < \lfloor (m+1)\beta \rfloor$. Clearly $\lfloor m\alpha \rfloor \leqslant \lfloor (m+1)\beta \rfloor$. Suppose $\lfloor m\alpha \rfloor = \lfloor (m+1)\beta \rfloor$. Then $\lfloor m\beta \rfloor + 1 = \lfloor (m+1)\beta \rfloor$. On the other hand, we have

$$\lfloor m\beta \rfloor + 2 \leqslant m\beta + 2 \leqslant m\beta + \beta = (m+1)\beta.$$

Thus $\lfloor (m+1)\beta \rfloor \geqslant \lfloor m\beta \rfloor + 2$, a contradiction. Therefore $\lfloor m\alpha \rfloor = \lfloor m\beta \rfloor + 1 < \lfloor (m+1)\beta \rfloor < \lfloor (m+1)\alpha \rfloor$ which again shows the claim. ■

Now we are going to study the above property in a general $\langle F, I \rangle$.

Definition 2.4 We say that $\langle F, I \rangle$ is *separable*, or $\langle F, I \rangle$ satisfies $\mathbb{S}$ property for short, if for every distinct $\alpha, \beta \geqslant 1$, $N_\alpha \neq N_\beta$ if and only if $\alpha \neq \beta$.

In Archimedean case, we prove $\mathbb{S}$ property by *induction*. In fact we show that if $\mathbb{N}_\alpha = \mathbb{N}_\beta$, then $\lfloor n\alpha \rfloor = \lfloor n_\beta \rfloor$ for all $n \in \mathbb{N}$. In non-Archimedean case, induction is too weak to can prove $\mathbb{S}$. M. Moniri has claimed that "*any arbitrary structure* $\langle F, I \rangle$ *is separable*", (private communication). We will show that a *weak* version of $\mathbb{S}$ property can be deduced in every $\langle F, I \rangle$. for this purpose, we need some auxiliary results. In theorem 2.2 (ii), we showed that $N_\alpha \cap N_\beta \neq \emptyset$ and $N_\alpha \cup N_\beta \neq I^{\geqslant 0}$ for rational elements. When α and β are irrational, the following result of Beatty [3], also reported in Skolem [11], provides a different view.

Theorem 2.5 *Let* α, β *be positive irrationals such that* $\alpha^{-1} + \beta^{-1} = 1$. *Then* $N_\alpha \cap N_\beta = \{0\}$ *and* $N_\alpha \cup N_\beta = I^{\geqslant 0}$.

Proof. To show $N_\alpha \cap N_\beta = \{0\}$, suppose there exists $0 \neq k \in N_\alpha \cap N_\beta$. Then there would be $m, n \in I^{>0}$ such that $k \leqslant m\alpha < m+1,\ k \leqslant n\beta < k+1$. Since α, β are irrationals, so the previous inequalities are proper. So

$$\frac{k}{m} < \alpha < \frac{k+1}{m},\ \frac{k}{n} < \beta < \frac{k+1}{n}. \qquad (1)$$

Thus $\frac{k}{n} < \frac{\alpha}{\alpha-1} < \frac{k+1}{n}$ where $\beta = \frac{\alpha}{\alpha-1}$. Hence $k(\alpha-1) < n\alpha < (k+1)(\alpha-1)$. But $\alpha, \beta > 1$, so $m, n < k$ and we have $(k-n)\alpha < k$ and $k+1 < (k+1-n)\alpha$. So

$$\frac{k+1}{k+1-n} < \alpha < \frac{k}{k-n}. \qquad (2)$$

By (1) and (2), we have $\frac{k}{m} < \frac{k}{k-n}$ and $\frac{k+1}{m} > \frac{k+1}{k+1-n}$, so $k-n < m < k+1-n$ implying $k < m+n < k+1$. This would be a contradiction, since $m, n, k \in I^{>0}$.

Next, we show $N_\alpha \cup N_\beta = I^{\geqslant 0}$. Suppose there is some $h \in I^{>0} \setminus (N_\alpha \cap N_\beta)$. Then there exist $m, n \in I^{\geqslant 0}$ such that $\lfloor m\alpha \rfloor < h < \lfloor (m+1)\alpha \rfloor$ and $\lfloor n\beta \rfloor < h < \lfloor (n+1)\beta \rfloor$ implying

$$\begin{aligned} m\alpha &< h &< (m+1)\alpha - 1, \\ n\beta &< h &< (n+1)\beta - 1. \end{aligned}$$

From these two we get $(h+1)\alpha^{-1} - 1 < m < h\alpha^{-1}$ and $(h+1)\beta^{-1} - 1 < n < h\beta^{-1}$. Therefore $(h+1)(\alpha^{-1} + \beta^{-1}) - 2 < m+n < h(\alpha^{-1} + \beta^{-1})$ and so $h+1-2 < m+n < h$ showing $h-1 < m+n < h$. Since $m, n \in I^{\geqslant 0}$, the last inequality is impossible. ■

We presented one direction of Fact A for real field in Theorem 2.5. For real case, the proof of the converse of Theorem 2.5 is based on **PHP** and some properties of an auxiliary function. We define $\mu(\alpha, h) = |\{n \in \mathbb{N} \mid \lfloor n\alpha \rfloor \leqslant h\}|$, the number of elements of $\mathbb{N}_\alpha$ not exceeding h. Note that in the real case, if $\mathbb{N}_\alpha \cup \mathbb{N}_\beta = \mathbb{N}$ and $\mathbb{N}_\alpha \cap \mathbb{N}_\beta = \emptyset$, then

$$\mu(\alpha, h) + \mu(\beta, h) = h. \qquad (*)$$

The $(*)$ equality provides the proof of the converse of Theorem 2.5 in the field of real numbers. We have $\lfloor \mu(\alpha, h)\alpha \rfloor \leqslant h < \lfloor (\mu(\alpha, h) + 1)\alpha \rfloor - 1$. Using this implication, we can define $\mu(\alpha, h)$ in non-Archimedean case. But this definition doesn't provide $(*)$ and consequently doesn't prove Fact A. On the other hand, if $\langle F, I \rangle$ has the $\mathbb{S}$ property, we can deduce Fact A as follows. Therefore study of the $\mathbb{S}$ property is a useful tool for extend the theory of Diophantine approximations to arbitrary ordered fields which equipped with integer parts.

Lemma 2.6 *Let $\langle F, I\rangle$ be separable. If $N_\alpha \cap N_\beta = 0$ and $N_\alpha \cup N_\beta = I^{\geqslant 0}$, then α, β are irrationals and $\alpha^{-1} + \beta^{-1} = 1$.*

Proof. By theorem 2.2, one of α, β is irrational. Suppose α is irrational. Set $\alpha^{-1} + \eta^{-1} = 1$. Then $N_\eta = N_\beta$ and $\eta, \beta > 1$. Since $\langle F, I\rangle$ is separable, $\eta = \beta$. ■

The following lemma is the rational version of Theorem 2.5. Its proof is exactly similar to theorem 2.5. The reader is refered to Theorem 3.15 in [9].

Lemma 2.7 *Let ρ, σ be two positive rationals such that $\rho^{-1} + \sigma^{-1} = 1$. Then*

$$\begin{aligned} N_\rho \cap N_\sigma &= \bigcup_{m \in \rho I \cap I^{\geqslant 0}} mI^{\geqslant 0} \\ I^{\geqslant 0} \setminus (N_\rho \cup N_\sigma) &= \bigcup_{m \in \rho I \cap I^{\geqslant 0}} (mI^{\geqslant 0} + (m-1)) \end{aligned}$$

Note that $\rho I \cap I^{\geqslant 0} = \sigma I \cap I^{\geqslant 0}$. If $\rho = \frac{k}{m}$ is such that $(m, k) = 1$, then $\sigma = \frac{k}{k-m}$, and we have $N_\rho \cap N_\sigma = kI^{\geqslant 0}$, and $I^{\geqslant 0} \setminus (N_\rho \cup N_\sigma) = kI^{\geqslant 0} + (k-1)$. If ρ has no irreducible representation, then $\rho I^{\geqslant 0} \cap I^{\geqslant 0}$ is union of arithmetical progressions of the form $m\rho I^{\geqslant 0}$ for some $m \in I^{>0}$ such that $m\rho \in I^{\geqslant 0}$. These arithmetical progressions have nonempty intersections, but it is impossible to find $k \in I^{\geqslant 0}$ such that $N_\rho \cap N_\sigma = kI^{\geqslant 0}$. The existence of an element like "k" is equivalent to the existence of an irreducible representation for ρ as a rational.

Now we can prove versions of the $\mathbb{S}$ property. The following theorem proves this property for some large classes.

Theorem 2.8 *Let $\alpha, \beta > 1$. Then $N_\alpha \neq N_\beta$ when*

1. *α, β are distinct irrationals;*

2. *α, β are distinct rationals.*

Proof. First, suppose α and β are distinct irrationals. It suffices to prove the lemma for $1 < \beta < \alpha < 2$. There exist η, γ such that $\eta^{-1} = 1 - \alpha^{-1}, \gamma^{-1} = 1 - \beta^{-1}$ and $2 < \eta < \gamma$. So $N_\gamma \neq N_\eta$ and there exists $x \in N_\eta \setminus N_\gamma$. Thus $x \in N_\beta \setminus N_\alpha$.

Now suppose $\rho, \sigma > 1$ are distinct rationals. Using Theorem 2.3, we have only to consider the case $\rho, \sigma < 2$. Let η and γ be rational elements such that $\rho^{-1} + \eta^{-1} = 1$ and $\sigma^{-1} + \gamma^{-1} = 1$.

Then $2 < \gamma < \eta$ and by the above lemma, we have

$$N_\rho = \left[I^{\geqslant 0} \setminus \left(N_\eta \cup \bigcup_{m \in \rho I \cap I^{\geqslant 0}} (mI^{\geqslant 0} + (m-1)) \right) \right] \cup (\rho I^{\geqslant 0} \cap I^{\geqslant 0}),$$

and

$$N_\sigma = \left[I^{\geqslant 0} \setminus \left(N_\gamma \cup \bigcup_{m \in \sigma I \cap I^{\geqslant 0}} (mI^{\geqslant 0} + (m-1)) \right) \right] \cup (\sigma I^{\geqslant 0} \cap I^{\geqslant 0}).$$

Consequently by theorem 2.3, there exists $x \in N_\gamma \setminus N_\eta$. Let $m \in (\gamma I^{\geqslant 0} \cap \eta I^{\geqslant 0} \cap I^{\geqslant 0})$ such that $x < m-1$. So $mI^{\geqslant 0} + x \subset N_\gamma \setminus N_\eta$. Then $mI^{\geqslant 0} + x \subset N_\rho \setminus N_\sigma$. ■

By the same method as used in Theorem 2.5, we have

Theorem 2.9 *Let $\alpha, \beta > 1$ be distinct irrationals and $a, b, c, d \in I^{\geqslant 0}$. Then the following properties hold*

1. *If $N_\alpha \cap N_\beta = \{0\}$ and $N_\alpha \cup N_\beta = I^{\geqslant 0}$, then $\alpha^{-1} + \beta^{-1} = 1$.*
2. *If $a\alpha^{-1} + b\beta^{-1} = 1$, then $N_\alpha \cap N_\beta = \{0\}$.*
3. *If $a(1-\alpha^{-1}) + b(1-\beta^{-1}) = 1$, then $N_\alpha \cup N_\beta = I^{\geqslant 0}$.*
4. *If $a\alpha^{-1} + b(1-\beta^{-1}) = 1$, then $N_\alpha \subseteq N_\beta$.*
5. *If $a\alpha^{-1} + b\beta^{-1} = 1, c(1-\alpha^{-1}) + d(1-\beta^{-1}) = 1$, then $a = b = c = d = 1$ (and so $\alpha^{-1} + \beta^{-1} = 1$).*

Proof. 1. This is obvious, since α and β are irrational.

2. Suppose there exists $0 \neq k \in (N_\alpha \cap N_\beta)$. Then there exist $m, n \in I^{>0}$ such that $k < m\alpha < k+1$ and $k < n\beta < k+1$. Consequently

$$\frac{k}{m} < \alpha < \frac{k+1}{m} \textit{and} \frac{k}{n} < \beta < \frac{k+1}{n} \ (\mathbf{1}).$$

Therefore $\frac{k}{m} < \frac{b\alpha}{\alpha - a} < \frac{k+1}{n}$, where $\beta = \frac{b\alpha}{\alpha - a}$. Hence $k(\alpha - a) < nb\alpha < (k+1)(\alpha - a)$ and so $(k - bn)\alpha < ak$ and $(k+1)a < (k+1-bn)\alpha$. We claim that $k - bn > 0$. To see this, suppose $\beta > \alpha$. Then $a\alpha^{-1} + b\beta^{-1} = 1$. Therefore $an\beta\alpha^{-1} + bn = n\beta$. So $an\beta\alpha^{-1} + bn < k+1$ and therefore $k+1-bn > 1$ implying $k - bn > 0$. Therefore we have $\frac{a(k+1)}{k+1-bn} < \alpha < \frac{ak}{k-bn}$. Using

(1), we have $\frac{k}{m} < \frac{ak}{k-bn}$ and $\frac{a(k+1)}{k+1-bn} < \frac{k+1}{m}$ and so $k < am - bn < k+1$. But $am - bn \in I$, a contradiction.

3. Let $\gamma^{-1} = 1 - \alpha^{-1}, \eta^{-1} = 1 - \beta^{-1}$. Then η and γ are irrationals and $a\gamma^{-1} + b\eta^{-1} = 1$. Therefore $N_\gamma \cap \ N_\eta = 0$. Since we have $N_\gamma \cap \ N_\alpha = 0$ and $N_\gamma \cup \ N_\alpha = I^{\geqslant 0}$, so $N_\eta \subseteq N_\alpha$ and finally $N_\beta \cup N_\alpha \supseteq N_\beta \cup N_\eta = I^{\geqslant 0}$.

4. Let $\eta^{-1} = 1 - \beta^{-1}$. Then $a\alpha^{-1} + b\eta^{-1} = 1$. Therefore $N_\eta \cap N_\alpha = 0$ and so $N_\alpha \subseteq N_\beta$.

5. We have $N_\alpha \cap N_\beta = 0$ and $N_\alpha \cup N_\beta = I^{\geqslant 0}$. Since α, β are irrationals, $\alpha^{-1} + \beta^{-1} = 1$. Therefore $a\alpha^{-1} + b\beta^{-1} = \alpha^{-1} + \beta^{-1}$. Hence we have $\alpha^{-1}(a-1) = \beta^{-1}(1-b) \geqslant 0$. From $b \geqslant 1$, we find that $b = a = 1$. Also, $c = d = 1$. ■

We now must determine the relation between N_α and N_ρ, when α is irrational and ρ is rational. Note that Q, the fraction field of I is a dense subfield of F and if F has an irrational element, Q and $F \setminus Q$ are proper dense subsets of F. So, if ρ is a rational element, then for each positive $\epsilon \in F$, there exist some irrationals α such that $|\alpha - \rho| < \epsilon$. Using this property, it is easy to define convergent sequences in scale of the ordered field F. Therefore if $cf(I) = \eta$, we have some η-sequences of irrationals which converge to ρ, (Note that $cf(F) = cf(I)$). The following considers this situation.

Theorem 2.10 *Let $\rho \geqslant 1$ be a rational. Suppose $cf(I) = \eta$, and $\{\alpha_\gamma\}_{\gamma<\eta}$ is a descending sequence of irrationals such that $\lim_{\gamma\to\eta} \alpha_\gamma = \rho$. Then for every $m \in I^{>0}$, there exists $\beta < \eta$ such that for all $\beta < \gamma < \eta$, we have $N_\rho|_{<m} = N_{\alpha_\gamma}|_{<m}$.*

Proof. Suppose $\rho = \frac{p}{q}$ is such that $p, q \in I^{>0}$. We can assume that $m = qt$. Otherwise, consider a multiple of q greater than m, such as $(\lfloor \frac{m}{q} \rfloor + 1)q$. Then $\rho = \frac{pt}{qt} = \frac{pt}{m}$ and therefore, for all $l \in I^{\geqslant 0}$ if $l \leqslant m$, there exists $u_l \in I^{\geqslant 0}$ such that $\rho \in [k + \frac{u_l}{l}, k + \frac{u_l+1}{l})$, where $k = \lfloor \rho \rfloor$. We show that there exists an interval which contains ρ in the intersection of $\bigcap_{l \leqslant m} [k + \frac{u_l}{l}, k + \frac{u_l+1}{l})$. For each interval $[k + \frac{u_l}{l}, k + \frac{u_l+1}{l})$, we have two cases: ($\rho = k + \frac{u_m}{m}$, $l < m$)

Case (1). $k + \frac{u_l}{l} \in [k + \frac{u_m}{m}, k + \frac{u_m+1}{m})$. In this case, we have $\rho = k + \frac{u_l}{l}$.

Case (2). $k + \frac{u_l+1}{l} \in [k + \frac{u_m}{m}, k + \frac{u_m+1}{m}]$. In fact, the two cases are disjoint, since the length of $[k + \frac{u_l}{l}, k + \frac{u_l+1}{l})$ is $\frac{1}{l} > \frac{1}{m}$. In case(2), we have

$$k + \frac{u_l+1}{l} - \rho = k + \frac{u_l+1}{l} - (k + \frac{u_m}{m}) = \frac{u_l+1}{l} - \frac{u_m}{m} \geqslant \frac{1}{ml} > \frac{1}{m^2}.$$

Therefore there exists an interval I with length at least $\frac{1}{m^2}$ such that for all l less than m, $I \subseteq [k + \frac{u_l}{l}, k + \frac{u_l+1}{l})$ contains ρ.

On the other hand, $\alpha_\gamma \searrow \rho$ and the α_γ's are irrationals. So there exists $\beta < \eta$ such that for all $\gamma > \beta$, we have $0 < \alpha_\gamma - \rho < \frac{1}{m^2}$ and therefore, for all $l \leqslant m$, $\alpha_\gamma \in [k + \frac{u_l}{l}, k + \frac{u_l+1}{l})$. So for all $\gamma > \beta$, $\alpha_\gamma \in I$. But for all $x \in I$ (such as ρ and α_γ for $\gamma > \beta$) and $l \leqslant m$, we have $k + \frac{u_l}{l} \leqslant x < k + \frac{u_l+1}{l}$ and so $kl + u_l \leqslant lx < kl + u_l + 1$ implying $\lfloor lx \rfloor = kl + u_l$. Since $km + u_m \geqslant m$, we get $\beta < \gamma < \eta$ and so $N_\rho|_{<m} = N_{\alpha_\gamma}|_{<m}$. ■

Below, using the above theorem, we generalize one direction of Fact F' without any condition and show that if I is a Bézout domain, Fact F' generalizes.

Theorem 2.11 *Suppose $\rho, \sigma > 1$ are rationals. Then we have*

1. *if there exist $a, b \in I^{\geqslant 0}$ such that $a\rho^{-1} + b(1 - \sigma^{-1}) = 1$. Then $N_\rho \subseteq N_\sigma$.*
2. *if I is Bézout and $N_\rho \subseteq N_\sigma$, then there exist $a, b \in I^{\geqslant 0}$ such that $a\rho^{-1} + b(1 - \sigma^{-1}) = 1$.*

Proof. 1. Let $\{\alpha_\eta\}$ be a decreasing sequence of irrationals which tends to σ. For every sufficiently large ordinal η, N_{α_η} and N_σ coincide on some initial segment with arbitrary large length. Choose $\{\beta_\eta\}$ such that $a(1 - \alpha_\eta)^{-1} + b\beta_\eta^{-1} = 1$. Then the sequence $\{\beta_\eta\}$ is a decreasing sequence converging to ρ. Now suppose $t \in N_\rho$. Then there is an ordinal γ such that for all $\eta > \gamma$, N_{β_η} and N_ρ coincide on $\leqslant t$. Note that $t \in N_{\beta_\eta}$. Since $N_{\beta_\eta} \subseteq N_{\alpha_\eta}$, we must have $t \in N_{\alpha_\eta}$. In particular, $t \in N_\alpha$.

2. We show that if $\sigma \in I^{>1}$, then $\rho \in I^{>0}$ and consequently ρ is a multiple of σ. Take $\rho = \frac{m}{n}$ with $(m, n) = 1$, $m > n > 1$. Since I is Bézout, there exist $s, t \in I$ such that $sm + tn = 1$. If $s < 0$, take $k \in I^{\geqslant 0}$ such that $k > [\frac{s}{n}]$. Then $s + kn \in I^{\geqslant 0}$ and $m(s + kn) + n(t - km) = 1$. Suppose $s > 0$. Then $ms = 1 + nk'$, $[s\rho] = [s \times \frac{m}{n}] = k'$, $[ns\rho] = ms$, and $(ms, k') = 1$. But ms, k' are both multiples of $\sigma \in I^{>0}$, a contradiction.

Now suppose $\sigma \in Q \setminus I$, $\sigma = \frac{m}{s}$ and let $\rho = \frac{m}{n}$ be such that $(m, n, s) = 1$. If $(m - s, n) = d$, then $\frac{m}{d}$ will be an integer multiple of both $\frac{m}{n}$ and $\frac{m}{m-s}$ (i.e., there will exist $u, v \in I$ such that $u \cdot \frac{m}{n} = \frac{m}{d}$ and $v \cdot \frac{m}{m-s} = \frac{m}{d}$). Then $N_{\frac{m}{d}} \subseteq N_{\frac{m}{m-s}}$ and $N_{\frac{m}{d}} \subseteq N_{\frac{m}{n}} \subseteq N_{\frac{m}{s}}$. So, $N_{\frac{m}{d}} \subseteq N_{\frac{m}{s}} \cap N_{\frac{m}{m-s}}$. But if $\eta = \frac{m}{m-s}$, then $\eta^{-1} + \sigma^{-1} = 1$ and if $k = \frac{m}{(m,s)}$, then $N_{\frac{m}{s}} \cap N_{\frac{m}{m-s}} = kI^{\geqslant 0}$. Therefore $N_{\frac{m}{d}} \subseteq N_k$. By the first paragraph, $d \mid m$ and $d \mid (m, n, s) = 1$. So, $(m - s, n) = (m, n, s) = 1$.

For all $j \in I^{\geqslant 0}$, there exists $x_j \in I^{\geqslant 0}$ such that $[j\rho] = [x_j\sigma]$. Thus $j\rho - y, xj\sigma - y$ have the same signs for all $y \in I$ (the sign of zero is taken here to be plus). Substituting ρ and σ, we conclude that $jm - yn$ and $x_jm - ys$ have the same signs for all j and y. We use absolute values of these numbers as a and b, with j and y chosen appropriately. First note that $a(1-\sigma^{-1}+b\rho^{-1}) =\mid (jm-yn)(1-\frac{m}{s}) + (x_jm-ys)\frac{n}{m} \mid=\mid j(m-s)+(x_j-y)n \mid$. For any fixed positive integer j, note that $x_j - y$ can assume all I-values. This argument proves the theorem unless one or the other of these values a or b is 0. But as the standard case (i.e., in the real field), we can choose appropriate a's and b's. ■

Corollary 2.12 *Suppose $\rho, \sigma > 1$ are rationals and I is Bézout. Then $N_\rho \subseteq N_\sigma$ if and only if there exist $a, b \in I^{\geqslant 0}$ such that $a\rho^{-1} + b(1-\sigma^{-1}) = 1$.*

Now we show that if $N_\sigma = N_\rho$, then σ, ρ are close to each other.

Lemma 2.13 *Let $\sigma, \rho \in F^{\geqslant 1}$ with $\sigma < \rho$ be such that $N_\sigma = N_\rho$. Then $\rho - \sigma$ is an infinitesimal.*

Proof. We already know that $1 \leqslant \sigma < \rho < 2$ and so $\lfloor \sigma \rfloor = \lfloor \rho \rfloor = 1$. Assume for the sake of a contradiction that $\rho - \sigma$ is not an infinitesimal and so its inverse is limited. Suppose $\rho - \sigma = r + \epsilon$ such that r is a real number and ϵ is an infinitesimal. Then $0 \leqslant r \leqslant 1$, (note that if $r = 1$ then $\epsilon < 0$). Thus $\frac{1}{\rho-\sigma} = \frac{1}{r+\epsilon}$ and $\frac{1}{r} - \frac{1}{r+\epsilon} = \frac{\epsilon}{r(r+\epsilon)}$ is an infinitesimal because r and $r+\epsilon$ are both finite. Therefore $\frac{1}{\rho-\sigma}$ has a standard integer part $m = \lfloor \frac{1}{\rho-\sigma} \rfloor$ which equals to $\lfloor \frac{1}{r} \rfloor$ or $\lfloor \frac{1}{r} \rfloor - 1$. Therefore m is **finite**. We have $(m+1)\sigma + 1 < (m+1)\rho$, since $\frac{1}{\rho-\sigma} < m+1$. Thus $\lfloor (m+1)\sigma \rfloor < \lfloor (m+1)\rho \rfloor$. So there exists $1 \leqslant k \leqslant m$ such that $\lfloor k\rho \rfloor = \lfloor k\sigma \rfloor$ and $\lfloor (k+1)\rho \rfloor > \lfloor (k+1)\sigma \rfloor$. Thus $\lfloor (k+1)\sigma \rfloor \in N_\sigma \setminus N_\rho$. ■

We prove that if $2 < \alpha < \beta$, then $\lfloor (m+1)\alpha \rfloor \in N_\alpha \setminus N_\beta$ for $m = \lfloor \frac{1}{\beta-\alpha} \rfloor$. Now let $1 < \alpha < \beta < 2$ and both are irrationals. So if $\alpha^{-1} + \eta^{-1} = 1$ and $\beta^{-1} + \gamma^{-1} = 1$, then $2 < \gamma < \eta$ and we have $\lfloor (m+1)\alpha \rfloor \in N_\gamma \setminus N_\eta$ for $m = \lfloor \frac{1}{\eta-\gamma} \rfloor$ and consequently it is in $N_\alpha \setminus N_\beta$. Note that $m = \lfloor \frac{(\alpha-1)(\beta-1)}{\beta-\alpha} \rfloor$. We use from this element to prove the $\mathbb{S}$ property for a suitable case.

Suppose $\rho < \beta$ and ρ is rational and β is irrational. Then for all irrationals sufficiently close to ρ (and greater than it), such as α, if $m = \lfloor \frac{(\alpha-1)(\beta-1)}{\beta-\alpha} \rfloor$, then we have $\lfloor (m+1)\frac{\beta}{\beta-1} \rfloor \in N_{\frac{\beta}{\beta-1}} \setminus N_{\frac{\alpha}{\alpha-1}}$. So, $\lfloor (m+1)\frac{\beta}{\beta-1} \rfloor \in N_\alpha \setminus N_\beta$ and thus

$$\frac{(\rho-1)(\beta-1)}{\beta-\rho} - \frac{(\alpha-1)(\beta-1)}{\beta-\alpha} = \frac{(\rho-1)(\alpha-\rho)}{(\beta-\rho)(\beta-\alpha)}(1-\beta) < 0.$$

So $\frac{(\rho-1)(\beta-1)}{\beta-\rho} < \frac{(\alpha-1)(\beta-1)}{\beta-\alpha}$ and we have $\lfloor (m+1)\frac{\rho}{\rho-1} \rfloor = \lfloor (m+1)\frac{\alpha}{\alpha-1} \rfloor$ for all irrationals α which are sufficiently closed to ρ (and greater than it). So $\lfloor (m+1)\frac{\alpha}{\alpha-1} \rfloor \in N_\rho \setminus N_\beta$.

If $\alpha < \rho < \beta$ and ρ is rational. We have $\frac{(\alpha-1)(\rho-1)}{\rho-\alpha} - \frac{(\alpha-1)(\beta-1)}{\beta-\alpha} = \frac{(\alpha-1)(\rho-\beta)}{(\rho-\alpha)(\beta-\alpha)} < 0$. So $\frac{(\alpha-1)(\rho-1)}{\rho-\alpha} < \frac{(\alpha-1)(\beta-1)}{\beta-\alpha}$. Fix $1 < \alpha < \rho < 2$, such that α is irrational and ρ is rational. If $m = \lfloor \frac{(\alpha-1)(\rho-1)}{\rho-\alpha} \rfloor$, then for all irrationals sufficiently close to ρ (and larger than it), called it β, we imply that $\lfloor (m+1)\frac{\beta}{\beta-1} \rfloor \in N_\alpha \setminus N_\beta$. We have $\frac{\rho}{\rho-1} - \frac{\beta}{\beta-1} = \frac{\beta-\rho}{(\beta-1)(\rho-1)} > 0$. So if β is sufficiently close to ρ, then $\lfloor (m+1)\frac{\rho}{\rho-1} \rfloor = \lfloor (m+1)\frac{\beta}{\beta-1} \rfloor$, unless $(m+1)\frac{\rho}{\rho-1} \in I^{>0}$. So, $\lfloor (m+1)\frac{\rho}{\rho-1} \rfloor \in N_\alpha \setminus N_\rho$, unless $(m+1)\frac{\rho}{\rho-1} \in I^{>0}$. If $(m+1)\frac{\rho}{\rho-1} \in I^{>0}$, then $(m+1)\frac{\rho}{\rho-1} \in N_\alpha \cap N_\rho \cap N_{\frac{\rho}{\rho-1}} \cap I^{>0}$.

Totally, if ρ is rational and β is irrational, we prove the $\mathbb{S}$ Property for all $1 < \rho < \beta < 2$ and for all $1 < \beta < \rho < 2$ s.t. $(m+1)\frac{\rho}{\rho-1} \notin I$ for $m = \lfloor \frac{(\beta-1)(\rho-1)}{\rho-\beta} \rfloor$. An example for the last case, in $\langle \mathbb{R}, \mathbb{Z} \rangle$, let $\rho = \frac{3}{2}$ and $\beta = \sqrt{2}$. Then $m = 2$, $(m+1)\frac{\rho}{\rho-1} = 9$, $7 \times \beta = \lfloor 7\sqrt{2} \rfloor = 9$ and $6 \times \rho = 9$.

Suppose $1 < \alpha < \beta < 2$ and α, β are two arbitrary elements of F such that $N_\alpha = N_\beta$. Then one of them is rational and the other is irrational. So for all $\gamma \in F$ which are $\alpha < \gamma < \beta$, $N_\gamma \neq N_\beta$. Because:

1. Let α be irrational and β be rational, if γ is irrational, $N_\gamma \neq N_\alpha$ and if γ is rational, $N_\gamma \neq N_\beta$.

2. Let α be rational and β be irrational, if γ is irrational, $N_\gamma \neq N_\beta$ and if γ is rational, $N_\gamma \neq N_\alpha$.

Now we define $\alpha \sim \beta$ if $N_\alpha = N_\beta$. This relation is an *equivalence relation* and we have:

(1) if $0 < \alpha \leqslant 1$, then $[\alpha]_\sim = (0, 1]_F$;

(2) if $\alpha \geqslant 2$, then $[\alpha]_\sim = \{\alpha\}$;

(3) If $\langle F, I \rangle$ is separable and $\alpha > 1$ then $[\alpha]_\sim = \{\alpha\}$;

(4) for an arbitrary $\langle F, I \rangle$, if $\alpha > 1$ then $[\alpha]_\sim = \{\alpha\}$ or $[\alpha]_\sim = \{\alpha, \beta\}$ such that if α is irrational then β is rational and vise versa and $\alpha - \beta$ is an infinitesimal element in F.

3 Arithmetical Progressions

In real case, for any irrational $\alpha > 0$, the set $\mathbb{N}_\alpha$ has a number of interesting number theoretic properties. For example, for each $k < m \in \mathbb{N}$, the subset $\{x|\ x \in \mathbb{N}_\alpha,\ x \equiv k\ (mod\ m)\}$ is unbounded and $\mathbb{N}_\alpha$ is uniformly distributed modulo of every $m \in \mathbb{N}$, [9]. In this section, we will show that the first property is always equivalent to the **DMO** property which is stronger than the $\mathbb{S}$ property.

3.1 P Condition

Definition 3.1 *A set $D \subset F$ is dense modulo one (or* **DMO***) with respect to I if the set $\{u - \lfloor u \rfloor_I | u \in D\}$ is dense in $[0,1)_F$.*

In [1], we presented some non-trivial **DMO** sets. Let recall one of those example.

Proposition 3.2 *For every $\langle F, I\rangle$ and $p \in \mathbb{N}$, the set $\{\sqrt[p]{u} \mid u \in I^{>0}\}$ is* **DMO** *with respect to every* **IP** *for F.*

Proof. Let I_1 be an **IP** for F. Suppose $k, t \in I_1^{\geqslant 0}$ and $k < t$. We need to find $M \in I$ and $n \in I_1$ such that $\frac{k}{t} < \sqrt[p]{M} - n < \frac{k+1}{t}$. We have $n + \frac{k}{t} < \sqrt[p]{M} < n + \frac{k+1}{t}$, equivalently, $(n+\frac{k}{t})^p < M < (n + \frac{k+1}{t})^p$. But $(n+\frac{k+1}{t})^p - (n+\frac{k}{t})^p \in Frac(I_1)$ and we have

$$(n+\frac{k+1}{t})^p - (n+\frac{k}{t})^p = \frac{1}{t}((n+\frac{k+1}{t})^{p-1} + (n+\frac{k+1}{t})^{p-2}(n+\frac{k}{t}) \ldots + (n+\frac{k}{t})^{p-1}).$$

Note that this is greater than $\frac{p}{t}(n+\frac{k}{t})^{p-1}$. So if we choose $n \in I_1$ such that the latter is greater than 1 (it suffices to choose $\lfloor \sqrt[p-1]{\frac{t}{p}} \rfloor + 1 \leqslant n$), then there will exist $M \in I^{>0}$ such that $n + \frac{k}{t} < \sqrt[p]{M} < n + \frac{k+1}{t}$. ■

Definition 3.3 *For an irrational $\alpha > 0$, we say that* **DMO**(α) *holds whenever the set $D = \{n\alpha | n \in I^{\geqslant 0}\}$ is* **DMO** *(with respect to I).*

Theorem 3.4 *If $1 \leqslant \alpha \in F$, the following are equivalent:*

(a) **DMO**(α),

(b) $(\forall m, k \in I^{\geqslant 0})(N_{m\alpha} \cap (m\ I^{\geqslant 0} + k) \neq \{0\})$.

Proof. $(\mathbf{a}) \to (\mathbf{b})$. By assumption, for all $k, m \in I^{\geqslant 0}$, there exists $u \in I^{\geqslant 0}$ such that $0 \leqslant \frac{k}{m} \leqslant u\alpha - \lfloor u\alpha \rfloor < \frac{k+1}{m} < 1$. This shows $k \leqslant mu\alpha - m\lfloor u\alpha \rfloor < k+1$ and so $m\lfloor u\alpha \rfloor + k \leqslant mu\alpha < m\lfloor u\alpha \rfloor + k + 1$ which in turn implies $\lfloor mu\alpha \rfloor \equiv k$, (mod m).

$(\mathbf{b}) \to (\mathbf{a})$. Suppose $0 < l < r < 1$. Since $Q = Frac(I)$ is a dense subfield of F, so there exist $p, q \in I^{\geqslant 0}$ such that $l < \frac{p}{q} < \frac{p+1}{q} < r$ (it suffices to assume $\frac{1}{q} < r - l$). By (b), there exists $n \in I^{\geqslant 0}$ such that $\lfloor nq\alpha \rfloor \equiv p \ (modq)$. Then $\exists t \in I^{\geqslant 0} qt + p \leqslant nq\alpha < qt + p + 1$ and so $t + \frac{p}{q} \leqslant n\alpha < t + \frac{p+1}{q}$. This implies $l < \frac{p}{q} \leqslant n\alpha - t < \frac{p+1}{q} < r$. Since $0 < n\alpha - t < 1$, $t = \lfloor n\alpha \rfloor$ and therefore $l < n\alpha - \lfloor n\alpha \rfloor < r$. ■

In the field of real numbers, if $\alpha > 1$ is irrational, then $\mathbb{N}_\alpha$ intersects any arithmetical progression, but does not contain any of them, (see [9, Theorem 3.3]). Therefore, Theorem 3.4 gives another proof for $\mathbf{DMO}(\alpha)$ in the standard situation. However we don't know whether $\mathbf{DMO}(\alpha)$ holds in general or not. In the real case, $\mathbf{DMO}(\alpha)$ for an irrational number α is usually obtained via cofinal rational quadratic approximations. We deal with this issue in Section 4. If $\mathbf{DMO}(\alpha)$ holds, then by Theorem 3.4 and Theorem 2.2, $\alpha > 1$ will be an irrational.

Corollary 3.5 *Suppose $\alpha > 1$ is irrational. Then the two conditions in Theorem 3.4 are equivalent to*

(c) $(\forall m \in I^{>0})(N_{m\alpha} \cap m\, I^{\geqslant 0} \neq \{0\})$.

Proof. The property $\mathbf{DMO}(\alpha)$ holds if and only if for all $\epsilon > 0$, there exists some $n \in I^{\geqslant 0}$ such that $n\alpha - \lfloor n\alpha \rfloor < \epsilon$. The reason goes as follows. Pick $0 < l < r < 1$ and let $\epsilon = r - l$. There exists $n \in I^{\geqslant 0}$ such that $n\alpha - \lfloor n\alpha \rfloor < \epsilon$. Therefore $\frac{r-l}{n\alpha - \lfloor n\alpha \rfloor} > 1$. So there exists $k \in I^{>0}$ such that $\frac{l}{n\alpha - \lfloor n\alpha \rfloor} < k < \frac{r}{n\alpha - \lfloor n\alpha \rfloor}$. Thus we have $0 < l < kn\alpha - k\lfloor n\alpha \rfloor < r < 1$ and therefore $k\lfloor n\alpha \rfloor = \lfloor kn\alpha \rfloor$. Now, let $m = kn$, and so $l < m\alpha - \lfloor m\alpha \rfloor < r$.

We have $b \to c$. Now let $\epsilon > 0$. Set $m \in I$ such that $m > \epsilon^{-1}$. Because of (c), there exists $k, t \in I^{>0}$, $mt < km\alpha < mt + 1$. Therefore $t < k\alpha < t + \frac{1}{m} < \epsilon$ and thus $0 < k\alpha - t < \epsilon$. Using the previous paragraph, proof is complete. ■

Now using the Theorem 3.4, we present a new property for structures $\langle F, I \rangle$:

Definition 3.6 Let $\alpha > 1$ be an irrational. We define four properties as follows:

$\mathbb{P}^1_\alpha$: The set N_α intersects each arithmetical progressions,

$\mathbb{P}^2_\alpha$: The set N_α does not contain any arithmetical progressions,

$\mathbb{P}$: For all irrationals $\alpha > 1$, $\mathbb{P}^1_\alpha$ holds,

$\mathbb{P}'$: For all irrationals $\alpha > 1$, $\mathbb{P}^2_\alpha$ holds.

Proposition 3.7 *If $\alpha, \beta > 1$ are irrationals such that $\beta^{-1} + \alpha^{-1} = 1$. Then*

(i) If $\mathbb{P}^2_\alpha$, then $(\forall m \in I^{>0})(\mathbb{P}^2_{m\alpha})$.

(ii) We have $(\forall m \in I^{>0})(\mathbb{P}^1_{m\alpha})$ if and only if $\mathbf{DMO}(\alpha)$.

(iii) The properties $\mathbb{P}^2_\alpha$ and $\mathbb{P}^1_\beta$ are equivalent.

Proof. (i) It is sufficient to observe that for all $m \in I^{\geqslant 0}$, we have $N_{m\alpha} \subseteq N_\alpha$.

(ii) This is just the content of Theorem 3.4.

(iii) Suppose that $\mathbb{P}^2_\alpha$ holds. Then N_α does not contain any arithmetical progressions. So N_β has a nonempty intersection with every arithmetical progression by Theorem 2.5. ■

The structure $\langle F, I\rangle$ satisfies **DMO** if F has irrational elements and for all irrational element α, $\mathbf{DMO}(\alpha)$ hold. Using the similar method, we have the following lemma.

Lemma 3.8 *Suppose that $\alpha > 1$ is irrational. Then we have the following:*

(i) If $N_\alpha \cup N_\gamma = I^{\geqslant 0}$ and $\mathbb{P}^2_\alpha$ holds, then so does $\mathbb{P}^1_\gamma$.

(ii) If $N_\alpha \cap N_\gamma = \{0\}$ and $\mathbb{P}^1_\gamma$ holds, then so does $\mathbb{P}^2_\alpha$.

(iii) The $\mathbb{P}$ property holds if and only if $\mathbb{P}'$ does.

(iv) The $\mathbb{P}$ property holds if and only if $(\forall \alpha > 1)$ with $\alpha \in F \setminus Q$, we have $\mathbf{DMO}(\alpha)$ or more continently $\langle F, I\rangle \models \mathbb{P}$ if and only if $\langle F, I\rangle \models \mathbf{DMO}$

In this section we study the structures $\langle F, I\rangle$ which satisfies the $\mathbb{P}$ property. By part (iv) of Lemma 3.8, this section is about **DMO**-$\langle F, I\rangle$, i.e. the structures $\langle F, I\rangle \vDash \mathbf{DMO}$. At first, by using the method similar to the above Lemma, we immediately get the following theorem.

Theorem 3.9 *Suppose that $\langle F, I\rangle$ satisfies the $\mathbb{P}$ property and $\alpha > 1$ is irrational and $\rho > 1$ is rational. Then neither of the relations below could hold:*

$N_\alpha \subseteq N_\rho$, $N_\rho \subseteq N_\alpha$, $N_\alpha \cap N_\rho = \{0\}$, $N_\alpha \cup N_\rho = I^{\geqslant 0}$.

Proof. Suppose $\rho = \frac{p}{q}$. Then $N_\rho \cap (pI^{\geqslant 0} + (p-1)) = \emptyset$. By the $\mathbb{P}$ property, N_α has a nontrivial intersection with this arithmetical progression. Hence $N_\alpha \nsubseteq N_\rho$.

The set N_ρ is a union of arithmetical progressions and $\mathbb{P}^2_\alpha$ holds. Therefore $N_\rho \nsubseteq N_\alpha$.

By $\mathbb{P}^1_\alpha$, we have $N_\rho \cap N_\alpha \neq \{0\}$.

We have $(pI^{\geqslant 0} + (p-1)) \setminus N_\alpha \neq \emptyset$ and $(pI^{\geqslant 0} + (p-1)) \cap N_\rho = \emptyset$. So, $N_\rho \cup N_\alpha \neq I^{\geqslant 0}$. ■

Using the above theorem, we introduce the relation between separability and the $\mathbb{P}$ property.

Corollary 3.10 *If $\langle F, I\rangle$ has the $\mathbb{P}$ property, then it is separable.*

Proof. We have shown already that $\langle F, I\rangle$ has the $\mathbb{S}$ property if and only if for every $1 \leqslant \beta < \alpha < 2$, with one of α, β being rational and the other irrational, $N_\alpha \neq N_\beta$. Theorem 3.9 completed the proof. ■

3.2 P Condition & Skolem-Bang's Theorems

It can be shown that the $\mathbb{P}$ property is a first order sentence in $\langle F, I\rangle$. So by Upward Löwenheim-Skolem theorem over $\langle \mathbb{R}, \mathbb{Z}\rangle$ (or over the countable structure $\langle \widetilde{\mathbb{Q}}, \mathbb{Z}\rangle$), there exist sufficiently large models of $\langle F, I\rangle \models \mathbb{P}$. Professor Moniri conjectures the following (private communication):

" $\langle F, I\rangle \models \mathbb{P}$, for all ordered field F with IP I."

But now we want to discuss about the $\mathbb{P}$ property and Skolem-Bang's Theorems. In this subsection, suppose $\langle F, I\rangle \models \mathbb{P}$.

Theorem 3.11 *If $I \models$ Béz and α, β are positive irrationals such that $1, \alpha, \beta$ are linearly dependent over the $Frac(I)$, say*

$$a\alpha + b\beta = c,\ (a, b, c) = 1 \text{ and } c > 1,$$

then the points $(m\alpha - \lfloor m\alpha \rfloor, m\beta - \lfloor m\beta \rfloor)_{m\in I^{>0}}$ *lie on, and only on, those portion of the lines* $ax + by = t$*, where* t *is any integer, lying within the unit square. Furthermore these points are dense on these segments.*

So we have the following corollaries by methods similar to which represented in [9, Section 3.5]:

Corollary 3.12 *(***Fact C***) If* $I \models B\acute{e}z$ *is an* **IP** *for* F *and suppose* α, β *are positive irrationals such that* $a\alpha^{-1} + b\beta^{-1} = c$ *for some* $a, b, c \in I$ *with* $ab < 0$ *and* $c \neq 0$*. Then* $N_\alpha \cap N_\beta$ *is a cofinal subset of* $I^{\geqslant 0}$*.*

Corollary 3.13 *(***Fact D***) Suppose* I *is a Bézout* **EDR** *and it is an* **IP** *for* F*. Let* α, β *be positive irrationals such that* $a\alpha^{-1} + b\beta^{-1} = c$ *for some* $a, b, c \in I^{>0}$ *with* $c > 1$ *and* $(a, b, c) = 1$*. Then* $N_\alpha \cap N_\beta$ *is a cofinal subset of* $I^{\geqslant 0}$*.*

We showed that if $a\alpha^{-1} + b\beta^{-1} = 1$, then $N_\alpha \cap N_\beta = 0$. So if $I \models B\acute{e}z$, the reminder case is

$$\{1, \alpha^{-1}, \beta^{-1}\} \text{ are linear independent over } Frac(I).$$

This case is *Kronecker's Theorem*. We don't know whether $\mathbb{P} \vdash Kronecker's\ Th.$ or not. If not, we must have some $\langle F, I\rangle \models \mathbb{P} + (\neg Kronecker's\ Th.)$.

4 Dirichlet's Theorem and Weak Fragments of Arithmetic

In this section, we prove the Dirichlet's Theorem and consequently the **DMO** property for a nontrivial structure $\langle F, I\rangle$. Classic proof of Dirichlet's Theorem is based on **PHP**. Using this fact, P. D'Aquino proved a weak version of this theorem, [5].

4.1 Weak PHP and Dirichlet's Theorem

P. D'Aquino studied the theory of Pell equation in $I\Delta_0$. She used a weak version of **PHP** which is called $\mathbf{\Delta_0} - \mathbf{WPHP}$:

$$for\ all\ x\ there\ is\ no\ 1-1\ \Delta_0 - function\ f\ such\ that\ f : 2x \longrightarrow x.$$

The principle $\mathbf{\Delta_0-WPHP}$ is available in the theory $I\Delta_0+\Omega_1$, where Ω_1 is

$$\forall x \exists y (x^{\lfloor \log_2 x \rfloor} = y).$$

The system $I\Delta_0+\Omega_1$ has been widely studied. We know that

$$IE_1 \subset IE_2 \subset \cdots I\Delta_0 \subsetneq I\Delta_0+\Omega_1.$$

P. D'Aquino proved the following version of Dirichlet's Theorem:

Theorem 4.1 *([5, Theorem 3.1]) Let $\mathcal{M} \vDash I\Delta_0+\Omega_1$, $d \in \mathcal{M}$, d not a square, $Q>1$, then there are $p,q \in \mathcal{M}$ such that $|p-\sqrt{d}q| < \frac{1}{Q}$, and $q<2Q$.*

We will prove a more strong version of Dirichlet's Theorem without *using* **PHP** or any weak version of it in the IE_1 system.

4.2 Farey Series And IE_1

First, we define *Farey series*. Then we prove some property of these series. Basic definitions and notations of this subsection are based on [6].

Definition 4.2 Suppose $I \vDash GCD$. For an arbitrary $N \in I^{>0}$, we can define *Farey series* $\mathfrak{F}_N$ of order N as follows. The Farey series $\mathfrak{F}_N$ is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed N. Thus $\frac{h}{k} \in \mathfrak{F}_N$ if $0 \leqslant h \leqslant k \leqslant N$, $(h,k)=1$.

We usually suppose $0 \in \mathfrak{F}_N$. Now, we prove some important properties of $\mathfrak{F}_N$.

Theorem 4.3 *Suppose $I \models GCD$. If $N \in I^{>0}$ and $0,1 \neq \frac{a}{b} \in \mathfrak{F}_N$ and exist $x_0, y_0 \in I$ such that $bx_0 - ay_0 = 1$, then*

1. *there exists unique successor for $\frac{a}{b}$ in $\mathfrak{F}_N$.*

   (i.e. there exists $\frac{c}{d} \in \mathfrak{F}_N$ such that $\frac{a}{b} < \frac{c}{d}$ and for all $\frac{a}{b} < \frac{m}{n} \in \mathfrak{F}_N$, $\frac{c}{d} \leqslant \frac{m}{n}$.)

2. *there exists unique pre-successor for $\frac{a}{b}$ in $\mathfrak{F}_N$.*

   (i.e. there exists $\frac{c}{d} \in \mathfrak{F}_N$ such that $\frac{a}{b} > \frac{c}{d}$ and for all $\frac{a}{b} > \frac{m}{n} \in \mathfrak{F}_N$, $\frac{c}{d} \geqslant \frac{m}{n}$.)

Proof. 1) Since (x_0, y_0) is a solution of $bx - ay = 1$, for each $r \in I$, $(x_0 + ra, y_0 + rb)$ is also a solution for $bx - ay = 1$. Choose r such that $N - b < y_0 + rb \leqslant N$, (we can do it by choose $r = \lfloor \frac{N-y_0}{k} \rfloor$). Now define $x = x_0 + rb$, $y = y_0 + ra$. Therefore, $N - b < y \leqslant N$, $bx = 1 + ay$. Thus $x = \frac{a}{b}y + \frac{1}{b} < y + \frac{1}{b}$. Then $(x, y) = 1, x \leqslant y$ and we have $\frac{x}{y} \in \mathfrak{F}_N$.

Note that $\frac{x}{y} = \frac{a}{b} + \frac{1}{ky} > \frac{a}{b}$. Consequently, $\frac{x}{y}$ appears after $\frac{a}{b}$ in $\mathfrak{F}_N$. If it is not successor of $\frac{a}{b}$, there exists some $\frac{h}{k}$ between $\frac{a}{b}$ and $\frac{x}{y}$. So we have

$$\frac{x}{y} - \frac{h}{k} = \frac{kx-hy}{ky} \geqslant \frac{1}{ky}$$
$$\frac{h}{k} - \frac{a}{b} = \frac{bh-ak}{bk} \geqslant \frac{1}{bk}.$$

On the other hand, we have

$$\frac{1}{by} = \frac{x}{y} - \frac{a}{b} = (\frac{x}{y} - \frac{h}{k}) + (\frac{h}{k} - \frac{a}{b}) \leqslant \frac{1}{ky} + \frac{1}{bk} = \frac{y+b}{bky}.$$

But $y + b > N$. Thus $\frac{x}{y} - \frac{a}{b} > \frac{N}{bky} \geqslant \frac{1}{by}$. It is a contradiction.

2) By the similarly method, we have $a(-y_0) - b(-x_0) = 1$. For all $r \in I$, $y = -y_0 + rb, x = -x_0 + ra$ is also a solution for $ay - bx = 1$. Now choose $r = \lfloor \frac{N+y_0}{b} \rfloor$. Then $\frac{x}{y} \in \mathfrak{F}_N$. We have $\frac{x}{y} < \frac{a}{b}$ and moreover $\frac{a}{b} - \frac{x}{y} = \frac{1}{by}$. By the similar inequalities, one can prove $\frac{x}{y}$ is a pre-successor of $\frac{a}{b}$. ■

It seems that the assumptions of Theorem 4.3 is essential, i.e. we have the following claim:

Claim Let I be a GCD domain and $N \in I^{>0}$. If there exists some $\frac{a}{b} \in \mathfrak{F}_N$ such that for all $x, y \in I$, $ax - by \neq 1$, then $\frac{a}{b}$ has no successor and pre-successor.

If $I \vDash B\acute{e}z$, then the assumptions of Theorem 4.3 are hold. So every element of $\mathfrak{F}_N$ which is not $0, 1$ has successor and pre-successor. For 0, we have the successor $\frac{1}{N}$ and for 1, we have the pre-successor $\frac{N-1}{N}$. We could prove some properties of $\mathfrak{F}_N$ for these integer parts:

Lemma 4.4 *If $I \models B\acute{e}z$, and $N \in I^{>0}$, then*

1. *If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive elements of $\mathfrak{F}_N$, then $k + k' > N$.*

2. *No two successive elements of $\mathfrak{F}_N$ has the same denominator.*

3. *If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive elements of $\mathfrak{F}_N$, then $kh' - hk' = 1$.*

Proof. 1) The mediant $\frac{h+h'}{k+k'}$ of $\frac{h}{k}$ and $\frac{h'}{k'}$, falls in the interval $(\frac{h}{k}, \frac{h'}{k'})$. So, if $k+k' \leqslant N$, then $\frac{h+h'}{k+k'}$ or the reduced format of it is in $\mathfrak{F}_N$ and it is between $\frac{h}{k}$ and $\frac{h'}{k'}$.

2) If $k>1$, and $\frac{h'}{k}$ succeeds $\frac{h}{k}$ in $\mathfrak{F}_N$, then $h+1 \leqslant h' < k$. But we have $hk < (h+1)(k-1)$, and therefore $\frac{h}{k-1} < \frac{h+1}{k}$. Then $\frac{h}{k} < \frac{h}{k-1} < \frac{h+1}{k} \leqslant \frac{h'}{k}$. But $\frac{h}{k-1}$ comes between $\frac{h}{k}$ and $\frac{h'}{k}$ in $\mathfrak{F}_N$, a contradiction.

3) Since $(h,k)=1$, the equation $kx-hy=1$ is soluble in I. If (x_0,y_0) is a solution, then (x_0+rh, y_0+rk) is also a solution for any $r \in I$. We can choose r so that $N-k < y_0+rk \leqslant N$. For this $N-k-y_0 < rk \leqslant N-y_0$. Then $\frac{N-y_0}{k} - 1 < r \leqslant \frac{N-y_0}{k}$. So $r = \lfloor \frac{N-y_0}{k} \rfloor$. Therefore, there is a solution (x,y) of the equation $kx-hy=1$ such that $(x,y)=1$ and

$$0 \leqslant N-k < y \leqslant N.$$

Note that $\frac{x}{y}$ is in its lowest terms and $y \leqslant N$ and we have $x = y\frac{h}{k} + \frac{1}{k} < y + \frac{1}{k}$. So $0 < x \leqslant y$. Thus $\frac{x}{y} \in \mathfrak{F}_N$. Also, we have $\frac{x}{y} = \frac{hy+1}{ky} = \frac{h}{k} + \frac{1}{ky} > \frac{h}{k}$. So that $\frac{x}{y}$ comes later in $\mathfrak{F}_N$ than $\frac{h}{k}$. If it is not $\frac{h'}{k'}$, it cames later than $\frac{h'}{k'}$ and $\frac{x}{y} - \frac{h'}{k'} = \frac{k'x-h'y}{yk'}$. Thus we have $\frac{x}{y} - \frac{h'}{k'} \geqslant \frac{1}{yk'}$.

While $\frac{h'}{k'} - \frac{h}{k} = \frac{kh'-hk'}{kk'}$, then $\frac{h'}{k'} - \frac{h}{k} \geqslant \frac{1}{kk'}$. Hence $\frac{1}{ky} = \frac{kx-hy}{ky}$, which equals to $\frac{x}{y} - \frac{h}{k}$. But it is less than or equals to $\frac{1}{yk'} + \frac{1}{kk'}$. The latter is equal to $\frac{k+y}{kk'y}$. But we have $N-k \leqslant y$, therefore $y+k > N$. Then $\frac{k+y}{kk'y} > \frac{N}{kk'y} \geqslant \frac{1}{ky}$. This is a contradiction and therefore $\frac{x}{y}$ must be $\frac{h'}{k'}$ and $kh'-hk'=1$. ■

Suppose that $I \models GCD$. For each $N \in I^{>0}$, we define a function $\varphi_N : \mathfrak{F}_N \to I^{>0}$ by $\varphi_N(\frac{a}{b}) = \lfloor N^2 \frac{a}{b} \rfloor$. φ_N is an embedding because if $\frac{a}{b} < \frac{c}{d}$ in $\mathfrak{F}_N$, then $\frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$. Since $bc-ad \in I^{>0}$, we have $\frac{bc-ad}{bd} \geqslant \frac{1}{bd}$. Moreover, $b,d \leqslant N$, then $\frac{c}{d} - \frac{a}{b} \geqslant \frac{1}{N^2}$. So $N^2\frac{a}{b} + 1 \leqslant N^2\frac{c}{d}$. Then $\varphi_N(\frac{a}{b}) = \lfloor N^2\frac{a}{b} \rfloor < \lfloor N^2 \frac{c}{d} \rfloor = \varphi_N(\frac{c}{d})$.

Lemma 4.5 *Suppose that $I \vDash IE_1$. For each $m \leqslant N^2$, there exists a greatest element of $\mathfrak{F}_N$ such as $\frac{x}{y}$, for which $\varphi_N(\frac{x}{y}) < m$.*

Proof. In fact, we show that $\{n \mid n \in I^{\geqslant 0},\ n \leqslant m, \exists \frac{x}{y} \in \mathfrak{F}_N, \varphi_N(\frac{x}{y}) = n\}$ is a nonempty bounded E_1-definable set. Note that $n = \varphi(\frac{x}{y})$ iff $ny \leqslant N^2x < (n+1)y$. Therefore $x < (n+1)\frac{y}{N^2} \leqslant \frac{n+1}{N}$. Thus we have $x < n$, as a weak inequality. On the other hand, $(x,y)=1$ is an E_1-definable sentence.

Now we define the following E_1-definable bounded subset with parameters N, m:

$$\exists x \leqslant mN,\ \exists 0 < y \leqslant N,\ x < y, (x, y) = 1\ \wedge\ ny \leqslant N^2 x < (n+1)y\ \wedge n \leqslant m.$$

The above bounded subset is nonempty, so it has a greatest element such as n_0, (see [12, (lemma 1.5)]). We have a unique element as $\frac{x}{y} \in \mathfrak{F}_N$ with respect to n_0. For this element, we have $\varphi_N(\frac{x}{y}) = n_0 \leqslant m$ and $\frac{x}{y}$ is a greatest element of $\mathfrak{F}_N$ for which this property holds. ■

Theorem 4.6 (Dirichlet's Approximation Lemma in IE_1)

Suppose that $I \models IE_1$. If $\alpha \in F \setminus Frac(I)$, for every $Q \in I^{>0}$, there exist $p, q \in I$ with $1 \leqslant q \leqslant Q$, $(p, q) = 1$ and $|\frac{p}{q} - \alpha| \leqslant \frac{1}{qQ}$.

Proof. It suffices to prove the result when $0 < \alpha < 1$. By Theorem 4.5, for $m = \lfloor Q^2\alpha \rfloor \leqslant Q^2$ there exists a greatest element of $\mathfrak{F}_Q$ such as $\frac{x}{y}$, for which $\varphi_Q(\frac{x}{y}) < m$. On the other hand $\frac{m}{Q^2} < \alpha < \frac{m+1}{Q^2}$ and there exists at most one element of $\mathfrak{F}_Q$ between $\frac{m}{Q^2}$ and $\frac{m+1}{Q^2}$. So the number α lies between two terms of the Farey series $\mathfrak{F}_Q$, say $\frac{p_1}{q_1} < \alpha < \frac{p_2}{q_2}$, (Note that $\frac{p_1}{q_1} = \frac{x}{y}$ obtained from Lemma 4.5 and $\frac{p_2}{q_2}$ obtained from part (1) of Theorem 4.3 as successor of $\frac{p_1}{q_1}$). Consider the mediant $\frac{p_1+p_2}{q_1+q_2}$; because this lies between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ and does not appear in $\mathfrak{F}_Q$, we must have $q_1 + q_2 \geqslant Q + 1$. Now α lies in one and only one of the intervals $(\frac{p_1}{q_1}, \frac{p_1+p_2}{q_1+q_2})$, $(\frac{p_1+p_2}{q_1+q_2}, \frac{p_2}{q_2})$.

If it lies in the first then, $|\alpha - \frac{p_1}{q_1}| \leqslant \frac{p_1+p_2}{q_1+q_2} - \frac{p_1}{q_1}$. The latter is equal to $\frac{p_2q_1-q_2p_1}{q_1(q_1+q_2)}$. Since $p_2q_1 - q_2p_1 = 1$, $\frac{p_2q_1-q_2p_1}{q_1(q_1+q_2)} = \frac{1}{q_1(q_1+q_2)}$. But we have $q_1 + q_2 \geqslant Q + 1$, so $|\alpha - \frac{p_1}{q_1}| \leqslant \frac{1}{q_1(Q+1)}$. Finally it is less than $\frac{1}{q_1Q}$, if we put $p = p_1$, $q = q_1$.

Similarly, if it lies in the second, then $|\alpha - \frac{p_2}{q_2}| \leqslant \frac{p_2}{q_2} - \frac{p_1+p_2}{q_1+q_2}$. The latter is equal to $\frac{p_2q_1-q_2p_1}{q_2(q_1+q_2)}$. Since $p_2q_1 - q_2p_1 = 1$, $\frac{p_2q_1-q_2p_1}{q_2(q_1+q_2)} = \frac{1}{q_2(q_1+q_2)}$. But we have $q_1 + q_2 \geqslant Q + 1$, so $|\alpha - \frac{p_2}{q_2}| \leqslant \frac{1}{q_2(Q+1)}$. Finally it is less than $\frac{1}{q_2Q}$ and we may take $p = p_2$, $q = q_2$. ■

The above format of Dirichlet's Theorem has some difference by Dirichlet's Theorem mentioned in Introduction. But we show that they are the same.

Corollary 4.7 *Suppose that $I \models IE_1$. If $\alpha \in F \setminus Frac(I)$, for every $Q \in I^{>0}$, there exist some $h, k \in I$ such that $k > Q$ and*

$$|\alpha - \frac{h}{k}| < \frac{1}{k^2}\ .$$

Proof. Suppose $0 < \alpha < 1$. First note that if $N_1 < N_2$, the fractions in the $\mathfrak{F}_{N_2} \setminus \mathfrak{F}_{N_1}$ have denominators larger than N_1.

Fix $Q \in I^{>0}$, and let $\frac{h}{k} \in \mathfrak{F}_Q$ be such that $|\alpha - \frac{h}{k}| < \frac{1}{kQ}$ and $k \leqslant Q$, certainly. Thus if $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive elements of $\mathfrak{F}_Q$ such that α lies between them, then set $\epsilon = min\{|\alpha - \frac{h}{k}|, |\alpha - \frac{h'}{k'}|\}$ and $N = \lfloor \frac{2}{\epsilon} \rfloor + 1$. It is obvious that $N > Q$. Now consider the set $\mathfrak{F}_N$. So by the first paragraph of proof, there exist some fractions of $\mathfrak{F}_N$ which lye certainly between α and $\frac{h}{k}$ and there exist some fractions of $\mathfrak{F}_N$ which lye certainly between α and $\frac{h'}{k'}$. These fractions have denominators greater than Q. Suppose $\frac{p}{q}$ and $\frac{p'}{q'}$ in $\mathfrak{F}_N$ such that α lies between them. Then the required inequality holds with $\frac{h}{k}$ replaced by at least one $\frac{p}{q}$, $\frac{p+p'}{q+q'}$ and $\frac{p'}{q'}$. ■

We can see Dirichlet's Approximation Lemma proves $\mathbb{P}$ property in IE_1-models, (see proposition 3.7(ii)). So the structures mentioned in theorem 4.5 are separable. Moreover Wilmers showed that $IE_1 \models B\acute{e}z$, [13]. So we have

Theorem 4.8 *Suppose that $I \models IE_1$. Then*

1. *(**Fact C**) If α, β be positive irrationals such that $a\alpha^{-1} + b\beta^{-1} = c$ for some $a, b, c \in I$ with $ab < 0$ and $c \neq 0$. Then $N_\alpha \cap N_\beta$ is a cofinal subset of $I^{\geqslant 0}$.*

2. *(**Fact D**) If α, β be positive irrationals such that $a\alpha^{-1} + b\beta^{-1} = c$ for some $a, b, c \in I^{>0}$ with $c > 1$ and $(a, b, c) = 1$. Then $N_\alpha \cap N_\beta$ is a cofinal subset of $I^{\geqslant 0}$.*

3. *(**Fact F'**) Suppose ρ and $\sigma > 1$ are rational. If $N_\rho \subseteq N_\sigma$, then there exist $a, b \in I^{\geqslant 0}$ such that $a\rho^{-1} + b(1 - \sigma^{-1}) = 1$.*

Corollary 4.7 provides a symmetric rational approximation for every irrational element α:

$$-\frac{1}{q^2} < \alpha - \frac{p}{q} < \frac{1}{q^2}.$$

In [10], B. Segre proved an asymmetric version of Dirichlet's Theorem. Niven presented a proof using Farey series, (see [9, Section 1.3]). In the rest of this section, we will show that this asymmetric Diophantine approximations Theorem holds for structures mentioned in theorem 4.6.

Applying proofs similar to the proof of Corollary 4.7, we conclude that if $r \in I$ is a positive element, then for all sufficiently large number n, the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ adjacent to α in $\mathfrak{F}_n$ have denominators larger than r, that is, $b > r$ and $d > r$. We begin with a preliminary result.

Lemma 4.9 *Let α be an irrational and $\tau > 0$. If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers with positive denominators such that $bc - ad = 1$ and*

$$\frac{a}{b} < \alpha < \frac{c}{d} .$$

Then the following inequalities holds with $\frac{h}{k}$ replaced by at least one of $\frac{a}{b}$, $\frac{a+c}{b+d}$ and $\frac{c}{d}$:

$$-\frac{1}{\sqrt{(1+4\tau)}k^2} < \alpha - \frac{h}{k} < \frac{\tau}{\sqrt{(1+4\tau)}k^2} .$$

Proof. The proof is similar to the proof Lemma 1.8 in [9]. ■

Now we can provide an asymmetric version of Dirichlet's Theorem.

Theorem 4.10 *Suppose that $I \models IE_1$. If $\alpha \in F \setminus Frac(I)$ and τ is an arbitrary positive element. For each element $Q \in I^{>0}$, there exists $h, k \in I$ such that $K > Q$ and*

$$-\frac{1}{\sqrt{(1+4\tau)}k^2} < \alpha - \frac{h}{k} < \frac{\tau}{\sqrt{(1+4\tau)}k^2} .$$

The proof is similar to the proof of [9], Theorem 1.7. So we only give one interesting corollary of this theorem.

Corollary 4.11 (Hurwitz's Approximation Lemma in IE_1)

Suppose that $I \models IE_1$. If $\alpha \in F \setminus Frac(I)$, for every $Q \in I^{>0}$, there exists $\frac{p}{q}$ with $q > Q$, $(p, q) = 1$ and $|\frac{p}{q} - \alpha| < \frac{1}{\sqrt{5}q^2}$.

Proof. It suffices to let $\tau = 1$ in the previous theorem. ■

Corollary 4.12 *Suppose that $I \models IE_1$. If $\alpha \in F \setminus Frac(I)$, for every $Q \in I^{>0}$, there exist $\frac{p_i}{q_i}$ with $q_i > Q$, $(p_i, q_i) = 1$ for $i = 1, 2$ such that $0 < \frac{p_1}{q_1} - \alpha < \frac{1}{q_1^2}$ and $0 < \alpha - \frac{p_2}{q_2} < \frac{1}{q_2^2}$.*

Proof. It suffices to let $\tau = 0$ for $\frac{p_1}{q_1}$ and replace α by $-\alpha$ for $\frac{P_2}{q_2}$. ■

5 Concluding Remarks and Questions

In this section, we mention some related questions and partial results.

5.1 Separable Fields

By the remark after Lemma 2.13, we proved the $\mathbb{S}$ property for a wide class of elements of an arbitrary structure $\langle F, I\rangle$. The remaining case is when

"β is irrational and ρ is ratioanl such that $1 < \beta < \rho < 2$ and $(m+1)\frac{\rho}{\rho-1} \in I$ for $m = \lfloor \frac{(\beta-1)(\rho-1)}{\rho-\beta} \rfloor$."

In this case, $(m+1)\frac{\rho}{\rho-1} \in N_\rho \cap N_\beta \cap N_{\frac{\rho}{\rho-1}}$. Therefore $k = \frac{m+1}{\rho-1} \in I$ and $k\rho = (m+1)\frac{\rho}{\rho-1}$. We have the following claim:

Claim- In the above case, $\lfloor (k+1)\rho \rfloor = k\rho + 1 \in N_\rho \setminus N_\beta$.

5.2 Kronecker's Theorem and Farey series

In the classical case, all implications of Theorem 2.9 are reversible (see [9]). Nevertheless, one can show that in the general $\langle F, I\rangle$ context, if the condition $N_\alpha \cap N_\beta = \{0\}$ implies the existence of $a, b \in I^{>0}$ with $a\alpha^{-1} + b\beta^{-1} = 1$, then all of the aforementioned implications are reversible. Furthermore, in this casde, the **DMO** property hold. These results depend on Kronecker's two dimensional **DMO** Theorem as appeared in [9]. It seems that the one dimensional **DMO** does not imply the two dimensional case. It is very interesting to prove Kronecker's Theorem without the assumption of PHP and only by using Farey series.

Question 5.1 *Does there exist any countable model $\langle F, +, \cdot, <, I\rangle$ satisfying* **DMO** *in which Kronecker's Theorem fails?*

We showed that the **DMO** property and the $\mathbb{P}$-condition are equivalent. Note that the **DMO** property is a first order sentence for $\langle F, +, \cdot, <, I\rangle$. So by the downward Löwenheim-Skolem theorem, it suffices to find out the answer to the following

Question 5.2 *Does* **DMO** *hold for all countable structures $\langle F, +, \cdot, <, I\rangle$?*

We showed in this paper that the $\mathbb{P}$-condition implies $\mathbb{S}$.

Question 5.3 *Can a model $\langle F, +, \cdot, <, I\rangle$ satisfy $\mathbb{S}$ but not the $\mathbb{P}$-condition?*

On the other hand, if we can prove the statement of Theorem 4.5 for $I \models B\acute{e}z$, then Dirichlet's Approximation Lemma will be proved for all $B\acute{e}z$ integer parts which can be the best result about Dirichlet's Approximation Lemma.

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